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## A Hierarchy of Super $w$ Strings

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### ABSTRACT

It is shown that the N-extended super  $w_n$  string can be obtained as a special class of vacua of both the N-extended super  $w_{n+1}$  string and the (N+1)-extended super  $w_n$  string. This hierarchy of super  $w$  string theories includes as subhierarchies both the superstring hierarchy given by Bastianelli, Ohta and Petersen and the  $w$  string hierarchy given by the authors.

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## 1. Introduction

The bosonic string theory can be regarded as a reparametrization invariant two dimensional field theory. We can similarly consider an extended string theory when the two dimensional field theory has larger local symmetry. Superstring theory is such an example that is invariant under the local supersymmetry. In this way, there are several string theories corresponding to their local symmetries. After fixing these gauge invariances to the (generalized) conformal gauge, these two dimensional field theories become conformal field theory with extended conformal symmetries. Therefore a string theory can be considered for each extension of the conformal algebra.

In the above sense, it has been found that a string theory can be viewed as a class of vacua for the other string theory, where the latter theory is based on a larger extended conformal algebra including the one of the former theory as a subalgebra. The first example of such a realization was studied by Berkovits and Vafa<sup>[1]</sup>. They showed that the bosonic ( $N = 1$ ) string theory can be viewed as a particular class of vacua for the  $N = 1$  ( $N = 2$ ) superstring, which can be explained by means of a spontaneous symmetry breaking of superconformal symmetry<sup>[2]</sup>. The larger superconformal symmetry is realized by introducing Nambu-Goldstone (NG) fields transformed inhomogeneously under the broken symmetry. This realization was generalized to general  $N$ -extended superstrings by Bastianelli, Ohta and Petersen<sup>[3]</sup>. They found that the  $N$ -extended superstring can be understood as a spontaneously broken phase of the  $(N + 1)$ -extended superstring and therefore there is a hierarchy of superstrings related by symmetry breaking.

It was also found that there are similar realizations for  $W$  string theories<sup>[4,5]</sup>. The bosonic string can be realized as a broken phase of the nonlinear  $W_3$  string<sup>[4]</sup> and it can be generalized to general ( $N = 1$ ) linear  $W_n$  strings by using the linear  $W_n$  algebra introduced in Ref. [6]<sup>[5]</sup>. We denote this linear  $W_n$  string simply by  $w_n$  string using lowercase  $w$ . This hierarchy of  $w$  strings has a different structure from the one of the superstrings since the  $w_n$  algebra is not a subalgebra of the  $w_{n+1}$

algebra. While the hierarchy relate the  $w_n$  string to the  $w_{n+1}$  string ( $n > 2$ ), both theories are in the broken phase. Only the bosonic ( $w_2$ ) string in this hierarchy can be in the general phase<sup>[5]</sup>.

In this paper, we construct a hierarchy of super  $w$  strings which is a natural generalization of the above two hierarchies. We show that general  $N$ -extended super ( $N$ -super)  $w_n$  string can be understood as a broken phase of both  $N$ -super  $w_{n+1}$  string and  $(N + 1)$ -super  $w_n$  string. The hierarchy is parametrized by two integers  $(N, n)$  and includes the superstring hierarchy  $(N, 2)$  and the  $w$  string hierarchy  $(0, n)$  as subhierarchies.

This paper is organized as follows.

Before constructing the super  $w$  string hierarchy, we give an algebraic definition of the  $N$ -super  $w_n$  string in Sec. 2. We introduce  $N$ -extended superspace ( $N$ -superspace) and write the  $N$ -super  $w_n$  algebra by using the  $N$ -extended superfields ( $N$ -superfields). Introducing the ghost  $N$ -superfields, the BRST charge is given by a  $N$ -superspace integral. For complete definition of the string theory, we must also give rules for calculating amplitudes which are not given in this paper. They are discussed in the final section. We also give the  $(N + 1)$ -super  $w_n$  string by using the  $N$ -superspace. This is useful for constructing the  $N$ -super  $w_n$  string as a broken phase of the  $(N + 1)$ -super  $w_n$  string explained in Sec. 4.

In Sec. 3, we construct such a realization of the  $N$ -super  $w_{n+1}$  string as is equivalent to the  $N$ -super  $w_n$  string. We first explain the results for the  $N = 0$  case obtained in the Ref. [5]. By introducing NG fields for each broken generator, the generators of the  $w_{n+1}$  algebra are constructed. The equivalence of cohomology is shown by finding the similarity transformation which maps the BRST charge of the  $w_{n+1}$  string to the sum of the one of the  $w_n$  string and a topological BRST charge. Since the cohomology of the topological BRST charge is trivial, the cohomology of two string theories coincide<sup>[7]</sup>. Then these results are extended to the general  $N$ -super  $w$  strings by using the  $N$ -superspace. This is a generalization of the  $w$  string hierarchy obtained in Ref. [5].

The  $N$ -super  $w_n$  string can also be obtained as a broken phase of the  $(N+1)$ -super  $w_n$  string, which is a generalization of the superstring hierarchy given in Ref. [3]. We consider it in Sec. 4 in a similar way to Sec. 3. The realization of the  $(N+1)$ -super  $w_n$  algebra is constructed by using the NG fields. The similarity transformation is found by separating it to three parts and guarantees the equivalence of the cohomology. By combining the results obtained in Sec. 3, we have a hierarchy of super  $w$  string parametrized two integers  $(N, n)$ . The  $N$ -super  $w_n$  string can be understood as a broken phase of both the  $N$ -super  $w_{n+1}$  string and the  $(N+1)$ -super  $w_n$  string. This super  $w$  string hierarchy is including both the superstring hierarchy<sup>[8]</sup>  $(N, 2)$  and the  $w$  string hierarchy<sup>[5]</sup>  $(0, n)$  as subhierarchies.

The Sec. 5 is devoted to discussion.

## 2. $N$ -super $w_n$ string

Before studying the super  $w$  string hierarchy, we give an algebraic definition of the  $N$ -super  $w_n$  string by constructing the BRST charge which determine physical states of the theory. We do not mention the rules of calculating amplitudes although they are necessary to the complete definition. We discuss it in the final section.

Operator product expansion (OPE) of the linear  $w_n$  algebra introduced in Ref. [6] has the form

$$W_i(z)W_j(w) \sim \frac{c(0, n)\delta_{i,0}\delta_{i,j}}{2(z-w)^4} + \frac{(i+j+2)W_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial W_{i+j}(w)}{z-w},$$

for  $i+j \leq n-2$

$$W_i(z)W_j(w) \sim 0, \quad \text{for } i+j > n-2 \quad (2.1)$$

where generators  $\{W_i(z); i = 0, \dots, n-2\}$  have dimensions  $i+2$  and  $W_0(z) = T(z)$  is the stress tensor. When the central charge is equal to the critical value  $c(0, n) = 2(n-1)(2n^2+2n+1)$ , we can define the  $w_n$  string theory by introducing fermionic

ghost fields  $(B_i(z), C_i(z))$  ( $i = 0, \dots, n-2$ ) with dimensions  $(i+2, -i-1)$ . The nilpotent BRST charge is defined by

$$Q_{BRST}(0, n) = \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} C_i (W_i + \frac{1}{2} W_i^{gh}),$$

where

$$W_i^{gh} = \sum_{j=0}^{n-i-2} (-(i+j+2) B_{i+j} \partial C_j - (j+2) \partial B_{i+j} C_j).$$

We extend this  $w_n$  string theory to the  $N$ -super ( $N \geq 1$ )  $w_n$  string theory.

We use the  $N$ -superspace to describe the  $N$ -super  $w_n$  algebra. The coordinates of the  $N$ -superspace are denoted by  $Z = (z, \theta^a)$  with  $a = 1, \dots, N$ . The other notations and conventions related to the superspace are <sup>[3]</sup>

$$\begin{aligned} D^a &= \frac{\partial}{\partial \theta^a} + \theta^a \partial, & \{D^a, D^b\} &= 2\delta^{a,b} \partial, \\ Z_{12} &= z_1 - z_2 - \theta_1^a \theta_2^a, & \Theta_{12}^a &= \theta_1^a - \theta_2^a, \\ \Theta_{12}^N &= \frac{1}{N!} \epsilon^{a_1 \dots a_N} \Theta_{12}^{a_1} \dots \Theta_{12}^{a_N}, & (\Theta_{12}^{N-1})^a &= \frac{1}{(N-1)!} \epsilon^{b_1 \dots b_{N-1} a} \Theta_{12}^{b_1} \dots \Theta_{12}^{b_{N-1}}, \\ \oint dZ &= \oint \frac{dz}{2\pi i} \int d\theta^N, & d\theta^N &= \frac{1}{N!} \epsilon^{a_1 \dots a_N} d\theta^{a_1} \dots d\theta^{a_N}. \end{aligned}$$

The  $N$ -super  $w_n$  algebra is generated by supercurrents  $\{\mathbf{W}_i(Z); i = 0, \dots, n-2\}$  which have dimensions  $i+2-N/2$  and  $\mathbf{W}_0(Z) = \mathbf{T}(Z)$  is the superstress tensor. The OPE defining the  $N$ -super  $w_n$  algebra is

$$\begin{aligned} \mathbf{W}_i(Z_1) \mathbf{W}_j(Z_2) &\sim (-1)^{N-1} \frac{\hat{c}(N, n) \delta_{i,0} \delta_{i,j}}{2Z_{12}^{4-N}} + \frac{\Theta_{12}^N}{Z_{12}^2} (i+j+2 - \frac{N}{2}) \mathbf{W}_{i+j}(Z_2) \\ &\quad + \frac{\Theta_{12}^N}{Z_{12}} (i+1) \partial \mathbf{W}_{i+j}(Z_2) + \frac{(\Theta_{12}^{N-1})^a}{Z_{12}} \frac{1}{2} D^a \mathbf{W}_{i+j}(Z_{12}), \\ &\hspace{25em} \text{for } i+j \leq n-2 \\ \mathbf{W}_i(Z_1) \mathbf{W}_j(Z_2) &\sim 0, \hspace{10em} \text{for } i+j > n-2 \end{aligned} \tag{2.2}$$

where the central extension term must vanish for  $N \geq 4$ . The critical central charge is obtained by calculating central charge of corresponding ghost fields and given by

$$\begin{aligned}\hat{c}(N, n) &= \frac{(3-N)!}{3!} c(N, n) \\ &= \begin{cases} (n-1)(2n+1) & \text{for } N=1 \\ (n-1) & \text{for } N=2 \\ 0 & \text{for } N \geq 3. \end{cases} \end{aligned} \quad (2.3)$$

Here we note that the critical central charge for  $N \geq 3$  is zero, which is consistent to the above argument of the central extension.

When the central charge is equal to this critical value, we can construct the  $N$ -super  $w_n$  string theory by introducing ghost fields  $(\mathbf{B}_i(Z), \mathbf{C}_i(Z))$  ( $i = 0, \dots, n-2$ ) satisfying

$$\mathbf{C}_i(Z_1) \mathbf{B}_j(Z_2) \sim \frac{\Theta_{12}^N}{Z_{12}} \delta_{i,j}.$$

The ghost superfields  $\mathbf{C}_i(Z)$  are fermionic with dimension  $(-i-1)$  and the anti-ghost superfields  $\mathbf{B}_i(Z)$  are fermionic (bosonic) for  $N = \text{even}$  (odd) with dimension  $(i+2-N/2)$ . The nilpotent BRST charge is given by the  $N$ -superspace integral as

$$Q_{BRST}(N, n) = \oint dZ \sum_{i=0}^{n-2} \mathbf{C}_i(\mathbf{W}_i + \frac{1}{2} \mathbf{W}_i^{gh}), \quad (2.4)$$

where

$$\mathbf{W}_i^{gh}(Z) = \sum_{j=0}^{n-i-2} \left( -(i+j+2 - \frac{N}{2}) \mathbf{B}_{i+j} \partial \mathbf{C}_j - (j+1) \partial \mathbf{B}_{i+j} \mathbf{C}_j - \frac{(-1)^N}{2} D^a \mathbf{B}_{i+j} D^a \mathbf{C}_j \right). \quad \blacksquare$$

The cohomology of this BRST charge describes physical states of the theory. Therefore we have defined, at least free, string theory. For calculating amplitudes, we must further study global degrees of freedom, which is discussed in the final section.

Before closing this section, we write the  $(N + 1)$ -super  $w_n$  string theory by using the  $N$ -superspace for later convenience. We restrict the central charge to the critical value (2.3) in the remaining part of this section.

We consider first the case of  $N = 0$ . The  $N = 1$   $w_n$  algebra is generated by two types of ordinary currents  $(W_i(z), V_i(z))$  ( $i = 0, \dots, n - 2$ ) with dimensions  $(i + 2, i + 3/2)$  as<sup>[5]</sup>

$$\begin{aligned} W_i(z)W_j(w) &\sim \frac{3(n-1)(2n+1)\delta_{i,0}\delta_{i,j}}{2(z-w)^4} + \frac{(i+j+2)W_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial W_{i+j}(w)}{z-w}, \\ W_i(z)V_j(w) &\sim \frac{(i+j+\frac{3}{2})V_{i+j}(w)}{(z-w)^2} + \frac{(i+1)\partial V_{i+j}(w)}{z-w}, \\ V_i(z)V_j(w) &\sim \frac{2(n-1)(2n+1)\delta_{i,0}\delta_{i,j}}{(z-w)^3} + \frac{2W_{i+j}(w)}{z-w}, \quad \text{for } i+j \leq n-2 \\ W_i(z)W_j(w) &\sim W_i(z)V_j(w) \sim V_i(z)V_j(w) \sim 0. \quad \text{for } i+j > n-2 \end{aligned}$$

We have to introduce both fermionic ghosts  $(B_i, C_i)$  and bosonic ghosts  $(\beta_i, \gamma_i)$  ( $i = 0, \dots, n - 2$ ) with dimensions  $(i + 2, -i - 1)$  and  $(i + 3/2, -i - 1/2)$ . The BRST charge  $Q_{BRST}(1, n)$  is rewritten as<sup>\*</sup>

$$Q_{BRST}(1, n) = \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} \left( C_i(W_i + \frac{1}{2}W_i^{gh}) + \gamma_i(V_i + \frac{1}{2}V_i^{gh}) \right),$$

where

$$\begin{aligned} W_i^{gh} &= W_i(BC) + W_i(\beta\gamma), \\ &= \sum_{j=0}^{n-i-2} \left( -(i+j+2)B_{i+j}\partial C_j - (j+1)\partial B_{i+j}C_j \right) \\ &\quad + \sum_{j=0}^{n-i-2} \left( -(i+j+\frac{3}{2})\beta_{i+j}\partial\gamma_j - (j+\frac{1}{2})\partial\beta_{i+j}\gamma_j \right), \\ V_i^{gh} &= \sum_{j=0}^{n-i-2} \left( -2B_{i+j}\gamma_j + (i+j+\frac{3}{2})\beta_{i+j}\partial C_j + (j+1)\partial\beta_{i+j}C_j \right). \end{aligned}$$

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<sup>\*</sup> We are taking a different convention from Ref[5] which are obtained by  $V_i^{(gh)} \rightarrow -V_i^{(gh)}$ .

For  $N \geq 1$ , the  $(N + 1)$ -super  $w_n$  algebra is rewritten by using two  $N$ -supercurrents  $(\mathbf{W}_i(Z), \mathbf{V}_i(Z))$  ( $i = 0, \dots, n - 2$ ) with dimensions  $(i + 2 - N/2, i + 3/2 - N/2)$ . The OPE can be written by using the  $N$ -superspace as

$$\begin{aligned}
\mathbf{W}_i(Z_1)\mathbf{W}_j(Z_2) &\sim \frac{(n-1)\delta_{N,1}\delta_{i,j}\delta_{i,0}}{Z_{12}^3} + \frac{\Theta_{12}^N}{Z_{12}^2}(i+j+2-\frac{N}{2})\mathbf{W}_{i+j}(Z_2) \\
&\quad + \frac{\Theta_{12}^N}{Z_{12}}(i+1)\partial\mathbf{W}_{i+j}(Z_2) + \frac{(\Theta_{12}^{N-1})^a}{Z_{12}}\frac{1}{2}D^a\mathbf{W}_{i+j}(Z_2), \\
\mathbf{W}_i(Z_1)\mathbf{V}_j(Z_2) &\sim \frac{\Theta_{12}^N}{Z_{12}^2}(i+j+\frac{3}{2}-\frac{N}{2})\mathbf{V}_{i+j}(Z_2) \\
&\quad + \frac{\Theta_{12}^N}{Z_{12}}(i+1)\partial\mathbf{V}_{i+j}(Z_2) + \frac{(\Theta_{12}^{N-1})^a}{Z_{12}}\frac{1}{2}D^a\mathbf{V}_{i+j}(Z_2), \\
\mathbf{V}_i(Z_1)\mathbf{V}_j(Z_2) &\sim -\frac{2(n-1)\delta_{N,1}\delta_{i,0}\delta_{i,j}}{Z_{12}^2} - \frac{\Theta_{12}^N}{Z_{12}}2\mathbf{W}_{i+j}(Z_2), \quad \text{for } i+j \leq n-2 \\
\mathbf{W}_i(Z_1)\mathbf{W}_j(Z_2) &\sim \mathbf{W}_i(Z_1)\mathbf{V}_j(Z_2) \sim \mathbf{V}_i(Z_1)\mathbf{V}_j(Z_2) \sim 0. \quad \text{for } i+j > n-2
\end{aligned} \tag{2.5}$$

The ghost superfields are  $(\mathbf{B}_i, \mathbf{C}_i)$  and  $(\beta_i, \gamma_i)$  ( $i = 0, \dots, n - 2$ ) with dimensions  $(i + 2 - N/2, -i - 1)$  and  $(i + 3/2 - N/2, -i - 1/2)$ , where ghost superfields  $\mathbf{C}_i$  are always fermionic and  $\gamma_i$  are always bosonic. On the other hand, anti-ghost superfields  $\mathbf{B}_i$  ( $\beta_i$ ) are fermionic (bosonic) for  $N = \text{even}$  and bosonic (fermionic) for  $N = \text{odd}$ . The BRST charge  $Q_{BRST}(N, n)$  is obtained by the  $N$ -superspace integral as

$$Q_{BRST}(N+1, n) = \oint dZ \sum_{i=0}^{n-2} \left( \mathbf{C}_i(\mathbf{W}_i + \frac{1}{2}\mathbf{W}_i^{gh}) + \gamma_i(\mathbf{V}_i + \frac{1}{2}\mathbf{V}_i^{gh}) \right),$$

where

$$\begin{aligned}
\mathbf{W}_i^{gh} &= \mathbf{W}_i(\mathbf{B}\mathbf{C}) + \mathbf{W}_i(\beta\gamma), \\
&= \sum_{j=0}^{n-i-2} \left( -\left(i+j+2-\frac{N}{2}\right)\mathbf{B}_{i+j}\partial\mathbf{C}_j - (j+1)\partial\mathbf{B}_{i+j}\mathbf{C}_j - \frac{(-1)^N}{2}D^a\mathbf{B}_{i+j}D^a\mathbf{C}_j \right) \\
&\quad + (-1)^N \sum_{j=0}^{n-i-2} \left( -\left(i+j+\frac{3}{2}-\frac{N}{2}\right)\beta_{i+j}\partial\gamma_j - \left(j+\frac{1}{2}\right)\partial\beta_{i+j}\gamma_j + \frac{(-1)^N}{2}D^a\beta_{i+j}D^a\gamma_j \right),
\end{aligned}$$



$$V_i^{gh} = \sum_{j=0}^{n-i-2} \left( 2\mathbf{B}_{i+j}\gamma_j + \left(i+j+\frac{3}{2}-\frac{N}{2}\right)\beta_{i+j}\partial\mathbf{C}_j + (j+1)\partial\beta_{i+j}\mathbf{C}_j - \frac{(-1)^N}{2}D^a\beta_{i+j}D^a\mathbf{C}_j \right).$$

### 3. The $N$ -super $w_n$ string as a broken phase of the $N$ -super $w_{n+1}$ string

In this section we generalize the  $w$  string hierarchy<sup>[5]</sup> to the  $N$ -super  $w$  string hierarchy with a fixed  $N$ . The  $N$ -super  $w_n$  string can be obtained as a broken phase of the  $N$ -super  $w_{n+1}$  string.

For making this paper self-contained, we first explain the results on the ( $N = 0$ )  $w$  string hierarchy obtained in Ref. [5]. The  $N = 0$   $w_{n+1}$  algebra can be realized by means of a general stress tensor  $T^{(m)}(z)$  with  $c = 26$  (unbroken generator) and the bosonic NG fields  $(\beta_i(z), \gamma_i(z))$  ( $i = 1, \dots, n-1$ ) with dimensions  $(i+2, -i-1)$ . The generators which satisfy the OPE (2.1) of the  $w_{n+1}$  algebra are written as

$$W_i(z) = \delta_{i,0}T^{(m)} + i\beta_i + \sum_{j=0}^{n-i-1} \left( -(i+j+2)\beta_{i+j}\partial\gamma_j - (j+1)\partial\beta_{i+j}\gamma_j \right).$$

We can find a similarity transformation by which the BRST charge (2.4)  $Q_{BRST}(0, n+1)$  is transformed into the sum of the BRST charges of the  $w_n$  string  $Q_{BRST}(0, n)$  and a topological BRST charge  $Q_t$ :

$$e^R Q_{BRST}(0, n+1) e^{-R} = Q_{BRST}(0, n) + Q_t, \quad Q_t = \oint \frac{dz}{2\pi i} (n-1) C_{n-1} \beta_{n-1},$$

$$R = \frac{1}{n-1} \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} C_i \left( (n+1) B_{n-1} \partial \gamma_{n-i-1} + (n-i) \partial B_{n-1} \gamma_{n-i-1} \right).$$

Since the cohomology of the topological BRST charge  $Q_t$  is trivial, the cohomology of  $Q_{BRST}(0, n+1)$  coincides with that of  $Q_{BRST}(0, n)$ . The physical spectrum of both theories are identical. Here we should note that the above

BRST charge  $Q_{BRST}(0, n)$  (or  $w_n$  algebra) is still written in terms of the NG fields  $(\beta_i(z), \gamma_i(z))$  ( $i = 1, \dots, n-2$ ). Thus the  $w_n$  string theory obtained in this way is in the broken phase. This is due to the fact that the  $w_n$  algebra is not a subalgebra of the  $w_{n+1}$  algebra. Only the  $w_2$  (bosonic) string can be in the unbroken phase. We should also note that, for completing the proof of the equivalence, we must further show the cancellation of zero modes since the string amplitudes do not consist of only BRST invariant quantities<sup>[1,5]</sup>. The BRST invariance holds only after integrations of the moduli.

Employing the  $N$ -superspace, we can easily generalize the above construction to the  $N$ -super  $w$  string hierarchy. The  $N$ -super  $w_{n+1}$  algebra can be realized by the general superstress tensor  $T^{(m)}(Z)$  with  $c = 15, 6$  and  $0$  for  $N = 1, 2$  and  $\geq 3$  and the NG superfields  $(\beta_i(Z), \gamma_i(Z))$  ( $i = 1, \dots, n-1$ ) with dimensions  $(i+2 - N/2, -i-1)$ .  $N$ -super  $w_{n+1}$  algebra is realized as

$$\begin{aligned} \mathbf{W}_i = & \delta_{i,o} \mathbf{T}^{(m)} + i \beta_i \\ & + \sum_{j=1}^{n-i-1} \left( (-1)^{N+1} (i+j+2 - \frac{N}{2}) \beta_{i+j} \partial \gamma_j + (-1)^{N+1} (j+1) \partial \beta_{i+j} \gamma_j + \frac{1}{2} D^a \beta_{i+j} D^a \gamma_j \right), \end{aligned} \quad (3.1)$$

where  $\beta_i$  are bosonic (fermionic) for  $N = \text{even}$  (odd) and  $\gamma_i$  are always bosonic superfields. The BRST charge  $Q_{BRST}(N, n+1)$  is transformed into the sum of  $Q_{BRST}(N, n)$  and a topological BRST charge  $Q_t$  by the similarity transformation as

$$\begin{aligned} e^R Q_{BRST}(N, n+1) e^{-R} &= Q_{BRST}(N, n) + Q_t, & Q_t &= (n-1) \oint dZ \mathbf{C}_{n-1} \beta_{n-1}, \\ R &= \frac{1}{n-1} \oint dZ \sum_{i=0}^{n-2} \mathbf{C}_i \left( (n+1 - \frac{N}{2}) \mathbf{B}_{n-1} \partial \gamma_{n-i-1} + (n-i) \partial \mathbf{B}_{n-1} \gamma_{n-i-1} \right. \\ &\quad \left. + \frac{(-1)^N}{2} D^a \mathbf{B}_{n-1} D^a \gamma_{n-i-1} \right). \end{aligned}$$

Discussions on the zero modes are given in the final section.

#### 4. The $N$ -super $w_n$ string as a broken phase of the $(N + 1)$ -super $w_n$ string

We consider a generalization of the superstring hierarchy<sup>[3]</sup> in this section. The  $N$ -super  $w_n$  string can be derived as a broken phase of the  $(N + 1)$ -super  $w_n$  string. We construct a special realization of the  $(N + 1)$ -super  $w_n$  algebra by adding NG superfields to generators of the  $N$ -super  $w_n$  algebra. It is shown that the cohomology of this  $(N + 1)$ - $w_n$  string coincides with the one of the  $N$ -super  $w_n$  string by finding a similarity transformation. Since  $N + 1 = 1, 2$ , and  $\geq 3$  are different in the critical central charge (2.3), we consider these cases separately.

##### 4.1. $N = 0$ AS $N = 1$

The  $N = 1$  super  $w_n$  algebra is realized by adding fermionic NG fields  $(\eta_i(z), \xi_i(z))$  ( $i = 0, \dots, n - 2$ ) to the generators of the  $w_n$  algebra  $W_i^{(m)}$  ( $i = 0, \dots, n - 2$ ) as<sup>★</sup>

$$\begin{aligned} W_i &= W_i^{(m)} + W_i(\eta\xi) + \delta_{i,0} \frac{n(n-1)}{4} \partial^2(\xi_0 \partial \xi_0), \\ V_i &= \eta_i + \sum_{j=0}^{n-i-2} \xi_j W_{i+j}^{(m)} + \sum_{\{i,j,k\}} (i+k+1) \xi_j \partial(\xi_k \eta_{i+j+k}) + \delta_{i,0} \frac{(2n+1)(n-1)}{2} \partial^2 \xi_0, \end{aligned} \quad (4.1)$$

where

$$W_i(\eta\xi) = \sum_{j=0}^{n-i-2} \left( -\left(i+j+\frac{3}{2}\right) \eta_{i+j} \partial \xi_j - \left(j+\frac{1}{2}\right) \partial \eta_{i+j} \xi_j \right),$$

and we use an abbreviated notation for multiple summations

$$\sum_{\{i,j,k,\dots\}} = \sum_{j=0}^{n-i-2} \sum_{k=0}^{n-i-j-2} \dots$$

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★ The realization (4.1) is different from the one given in Ref. [5].

It is useful to separate the similarity transformation to three parts<sup>[3]</sup>:

$$\begin{aligned}
R_1 &= \oint \frac{dz}{2\pi i} \sum_{\{0;i,j\}} B_{i+j} \gamma_i \xi_j, \\
R_2 &= - \oint \frac{dz}{2\pi i} \sum_{\{0;i,j,k\}} (j + \frac{1}{2}) \beta_{i+j+k} \gamma_i \xi_j \partial \xi_k, \\
R_3 &= \oint \frac{dz}{2\pi i} \sum_{\{0;i,j\}} C_i ((i+j + \frac{3}{2}) \beta_{i+j} \partial \xi_j + (j + \frac{1}{2}) \partial \beta_{i+j} \xi_j).
\end{aligned}$$

These similarity transformations map the BRST charge  $Q_{BRST}(1, n)$  to the sum of the BRST charge  $Q_{BRST}(0, n)$  and a topological charge  $Q_t$  as

$$e^{R_1} Q_{BRST}(1, n) e^{-R_1} = Q_1, \quad e^{R_2} Q_1 e^{-R_2} = Q_2, \quad e^{R_3} Q_2 e^{-R_3} = Q_{BRST}(0, n) + Q_t, \blacksquare$$

where

$$\begin{aligned}
Q_1 &= \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} \left( C_i(W_i^{(m)} + \frac{1}{2} W_i(BC) + W_i(\beta\gamma) + W_i(\eta\xi) + \delta_{i,0} \frac{n(n-1)}{4} \partial^2(\xi_0 \partial \xi_0)) \right. \\
&\quad + \gamma_i (\eta_i - \sum_{\{i;j,k\}} (j + \frac{1}{2}) \eta_{i+j+k} \xi_j \partial \xi_k - \delta_{i,0} \frac{n(n-1)}{4} \xi_0 \partial^2(\xi_0 \partial \xi_0)) \\
&\quad \left. - \sum_{j=0}^{n-i-2} \gamma_i \xi_j W_{i+j}(\beta\gamma) \right), \\
Q_2 &= \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} \left( C_i(W_i^{(m)} + \frac{1}{2} W_i(BC) + W_i(\beta\gamma) + W_i(\eta\xi)) + \gamma_i \eta_i \right), \\
Q_{BRST}(0, n) &= \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} C_i(W_i^{(m)} + \frac{1}{2} W_i(BC)), \quad Q_t = \oint \frac{dz}{2\pi i} \sum_{i=0}^{n-2} \gamma_i \eta_i.
\end{aligned}$$

These results reproduce those obtained in Ref. [8] for  $N$ -super  $w_2$  string  $(N, 2)$ .

#### 4.2. $N = 1$ AS $N = 2$

Using NG ( $N = 1$ ) superfields  $(\boldsymbol{\eta}_i(Z), \boldsymbol{\xi}_i(Z))$  ( $i = 0, \dots, n-2$ ) and the generators of the  $N = 1$  super  $w_n$  algebra  $\mathbf{W}_i^{(m)}$  ( $i = 0, \dots, n-2$ ), the  $N = 2$  super  $w_n$  algebra is realized by the following infinite series<sup>[8]</sup>

$$\begin{aligned} \mathbf{W}_i &= \mathbf{W}_i^{(m)} + \mathbf{W}_i(\boldsymbol{\eta}\boldsymbol{\xi}) + \delta_{i,0}(n-1) \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k \partial(\boldsymbol{\xi}_0 \partial D \boldsymbol{\xi}_0 (D \boldsymbol{\xi}_0)^{2k-2}), \\ \mathbf{V}_i &= \boldsymbol{\eta}_i - \sum_{j=0}^{n-i-2} \boldsymbol{\xi}_j \mathbf{W}_{i+j}^{(m)} \\ &\quad + \sum_{\{i;j,k\}} \left( (i+k+\frac{1}{2}) \boldsymbol{\xi}_j \partial(\boldsymbol{\xi}_k \boldsymbol{\eta}_{i+j+k}) + \frac{1}{2} D \boldsymbol{\eta}_{i+j+k} \boldsymbol{\xi}_j D \boldsymbol{\xi}_k + \frac{1}{4} \boldsymbol{\eta}_{i+j+k} D \boldsymbol{\xi}_j D \boldsymbol{\xi}_k \right) \\ &\quad + \delta_{i,0}(n-1) \left( \partial D \boldsymbol{\xi}_0 + \sum_{k=1}^{\infty} \left(\frac{-1}{4}\right)^k D(\boldsymbol{\xi}_0 \partial D \boldsymbol{\xi}_0 (D \boldsymbol{\xi}_0)^{2k-1}) \right), \end{aligned}$$

where

$$\mathbf{W}_i(\boldsymbol{\eta}\boldsymbol{\xi}) = \sum_{j=0}^{n-i-2} \left( -(i+j+1) \boldsymbol{\eta}_{i+j} \partial \boldsymbol{\xi}_j - (j+\frac{1}{2}) \partial \boldsymbol{\eta}_{i+j} \boldsymbol{\xi}_j + \frac{1}{2} D \boldsymbol{\eta}_{i+j} D \boldsymbol{\xi}_j \right).$$

and  $\boldsymbol{\eta}_i$  are bosonic with dimensions  $i + 3/2$  and  $\boldsymbol{\xi}_i$  are fermionic with dimensions  $-i - 1$ .

The similarity transformation is now

$$\begin{aligned} R_1 &= - \oint dZ \sum_{\{0;i,j\}} \mathbf{B}_{i+j} \boldsymbol{\gamma}_i \boldsymbol{\xi}_j, \\ R_2 &= \oint dZ \sum_{\{0;i,j,k\}} \left[ \boldsymbol{\gamma}_i \left( \frac{1}{2} (i+k+\frac{1}{2}) \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j \partial \boldsymbol{\xi}_k - \frac{1}{4} D \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j D \boldsymbol{\xi}_k \right. \right. \\ &\quad \left. \left. + \frac{1}{8} \boldsymbol{\beta}_{i+j+k} D \boldsymbol{\xi}_j D \boldsymbol{\xi}_k - \frac{1}{4} (j-k) \partial \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j \boldsymbol{\xi}_k \right) \right. \\ &\quad \left. + \boldsymbol{\eta}_{i+j+k} \left( -\frac{1}{4} (j-k) \partial \boldsymbol{\xi}_i \boldsymbol{\xi}_j \boldsymbol{\xi}_k - \frac{1}{8} \boldsymbol{\xi}_i D \boldsymbol{\xi}_j D \boldsymbol{\xi}_k \right) \right], \end{aligned}$$

$$R_3 = - \oint dZ \sum_{\{0;i,j\}} C_i \left( (i+j+1)\beta_{i+j}\partial\xi_j + (j+\frac{1}{2})\partial\beta_{i+j}\xi_j + \frac{1}{2}D\beta_{i+j}D\xi_j \right),$$

which transform the BRST charge  $Q_{BRST}(2, n)$  as

$$e^{R_1}Q_{BRST}(2, n)e^{-R_1} = Q_1, \quad e^{R_2}Q_1e^{-R_2} = Q_2, \quad e^{R_3}Q_2e^{-R_3} = Q_{BRST}(1, n) + Q_t,$$

$$Q_1 = \tilde{Q}_1 + \sum_{k=1}^{\infty} O_k,$$

$$\begin{aligned} \tilde{Q}_1 = \oint dZ \left( \sum_{i=0}^{n-2} \left[ C_i(\mathbf{W}_i^{(m)} + \frac{1}{2}\mathbf{W}_i(BC) + \mathbf{W}_i(\beta\gamma) + \mathbf{W}_i(\eta\xi)) \right. \right. \\ \left. \left. + \gamma_i \left( \eta_i + \sum_{\{i;j,k\}} \left( - (j+\frac{1}{2})\eta_{i+j+k}\xi_j\partial\xi_k + \frac{1}{4}\eta_{i+j+k}D\xi_jD\xi_k \right) \right) \right] \right. \\ \left. + \sum_{\{0;i,j\}} \gamma_i\xi_j\mathbf{W}_{i+j}(\beta\gamma) - \frac{(n-1)}{4}C_0\partial(\xi_0\partial D\xi_0) \right), \end{aligned}$$

$$O_k = (n-1)\frac{(-1)^k}{4} \left( \gamma_0\xi_0\partial(\xi_0\partial D\xi_0(D\xi_0)^{2k-2}) + \gamma_0D(\xi_0\partial D\xi_0(D\xi_0)^{2k-1}) \right. \\ \left. - C_0\partial(\xi_0\partial D\xi_0(D\xi_0)^{2k}) \right),$$

$$Q_2 = \oint dZ \sum_{i=0}^{n-2} \left[ C_i(\mathbf{W}_i^{(m)} + \frac{1}{2}\mathbf{W}_i(BC) + \mathbf{W}_i(\beta\gamma) + \mathbf{W}_i(\eta\xi) + \gamma_i\eta_i \right],$$

$$Q_{BRST}(1, n) = \oint dZ \sum_{i=0}^{n-2} C_i(\mathbf{W}_i^{(m)} + \frac{1}{2}\mathbf{W}_i(BC)), \quad Q_t = \oint dZ \sum_{i=0}^{n-2} \gamma_i\eta_i.$$

Here we should note the similarity transformation generated by  $R_2$ . It is convenient to consider the inverse relation since similarity transformation is invertible:

$$\begin{aligned} e^{-R_2}Q_2e^{R_2} &= \frac{1}{k!} \sum_{k=0}^{\infty} \underbrace{[-R_2, [\cdots, [-R_2, Q_2] \cdots]]}_k \\ &= Q_1. \end{aligned} \tag{4.2}$$

Except for this transformation (4.2), the right hand side (r.h.s.) of similar relations is terminated at finite order for all the other similarity transformations. Thus all

the other equations in this paper can be confirmed by brute force calculation. For this  $R_2$  transformation, however, r.h.s. is not terminated but it maps the finite series  $Q_2$  to the infinite series  $Q_1$ . This can be shown by using induction as follows. First we can show

$$\begin{aligned} Q_2 + [-R_2, Q_2] &= \tilde{Q}_1, \\ \frac{1}{2}[-R_2, [-R_2, Q_2]] &= \frac{1}{2}[-R_2, \tilde{Q}_1 - Q_2] \\ &= O_1. \end{aligned}$$

This is the starting point of the induction. Then suppose

$$\frac{1}{(k+1)!} \underbrace{[-R_2, [\cdots, [-R_2, Q_2] \cdots]]}_{k+1} = O_k.$$

It can be shown

$$\begin{aligned} \frac{1}{(k+2)!} \underbrace{[-R_2, [\cdots, [-R_2, Q_2] \cdots]]}_{k+2} &= \frac{1}{k+2} [-R_2, O_k] \\ &= O_{k+1}. \end{aligned}$$

Therefore the induction is completed.

#### 4.3. $N$ AS $N + 1$ FOR $N \geq 2$

For  $N \geq 2$ , all the structures are common since the critical central charges for these cases are zero (2.3). We introduce NG  $N$ -superfields  $(\boldsymbol{\eta}_i, \boldsymbol{\xi}_i)$ . While  $\boldsymbol{\xi}_i$  are always fermion with dimension  $-i - 1/2$ ,  $\boldsymbol{\eta}_i$  are fermion (boson) for  $N = \text{even}$  (odd) with dimension  $i + 3/2 - N/2$ . By means of these NG fields and the generators of the  $N$ -super  $w_n$  algebra,  $\mathbf{W}_i^{(m)}$  ( $i = 0, \cdots, n - 2$ ), we can construct generators of the  $(N + 1)$ -super  $w_n$  algebra as follows.

$$\mathbf{W}_i = \mathbf{W}_i^{(m)} + \mathbf{W}_i(\boldsymbol{\eta}\boldsymbol{\xi})$$

$$\begin{aligned}
&= \mathbf{W}_i^{(m)} + \sum_{j=0}^{n-i-2} \left( -\left(i+j+\frac{3}{2}-\frac{N}{2}\right) \boldsymbol{\eta}_{i+j} \partial \boldsymbol{\xi}_j - \left(j+\frac{1}{2}\right) \partial \boldsymbol{\eta}_{i+j} \boldsymbol{\xi}_j - \frac{(-1)^N}{2} D^a \boldsymbol{\eta}_{i+j} D^a \boldsymbol{\xi}_j \right), \\
V_i &= \boldsymbol{\eta}_i - \sum_{j=0}^{n-i-2} \boldsymbol{\xi}_j \mathbf{W}_{i+j}^{(m)} \\
&+ (-1)^N \sum_{\{i;j,k\}} \left( -\left(i+k+1-\frac{N}{2}\right) \boldsymbol{\xi}_j \partial (\boldsymbol{\xi}_k \boldsymbol{\eta}_{i+j+k}) + \frac{(-1)^N}{2} D^a \boldsymbol{\eta}_{i+j+k} \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k \right. \\
&\quad \left. - \frac{1}{4} \boldsymbol{\eta}_{i+j+k} D^a \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k \right).
\end{aligned}$$

The similarity transformation is given by three succeeded transformations:

$$\begin{aligned}
R_1 &= - \oint dZ \sum_{\{0;i,j\}} \mathbf{B}_{i+j} \boldsymbol{\gamma}_i \boldsymbol{\xi}_j, \\
R_2 &= \oint dZ \sum_{\{0;i,j,k\}} \left[ \boldsymbol{\gamma}_i \left( \frac{1}{2} \left( i+k+1-\frac{N}{2} \right) \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j \partial \boldsymbol{\xi}_k + \frac{(-1)^N}{4} D^a \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k \right. \right. \\
&\quad \left. \left. + \frac{1}{8} \boldsymbol{\beta}_{i+j+k} D^a \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k - \frac{1}{4} (j-k) \partial \boldsymbol{\beta}_{i+j+k} \boldsymbol{\xi}_j \boldsymbol{\xi}_k \right) \right. \\
&\quad \left. + (-1)^N \boldsymbol{\eta}_{i+j+k} \left( \frac{1}{4} (j-k) \partial \boldsymbol{\xi}_i \boldsymbol{\xi}_j \boldsymbol{\xi}_k + \frac{1}{8} \boldsymbol{\xi}_i D^a \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k \right) \right], \\
R_3 &= (-1)^N \oint dZ \sum_{\{0;i,j\}} \mathbf{C}_i \left( \left( i+j+\frac{3}{2}-\frac{N}{2} \right) \boldsymbol{\beta}_{i+j} \partial \boldsymbol{\xi}_j + \left( j+\frac{1}{2} \right) \partial \boldsymbol{\beta}_{i+j} \boldsymbol{\xi}_j \right. \\
&\quad \left. - \frac{(-1)^N}{2} D^a \boldsymbol{\beta}_{i+j} D^a \boldsymbol{\xi}_j \right),
\end{aligned}$$

which transform the BRST charge  $Q_{BRST}(N+1, n)$  as

$$e^{R_1} Q_{BRST}(N+1, n) e^{-R_1} = Q_1, \quad e^{R_2} Q_1 e^{-R_2} = Q_2, \quad e^{R_3} Q_2 e^{-R_3} = Q_{BRST}(1, n) + Q_t,$$

$$\begin{aligned}
Q_1 &= \oint dZ \left( \sum_{i=0}^{n-2} \left[ \mathbf{C}_i \left( \mathbf{W}_i^{(m)} + \frac{1}{2} \mathbf{W}_i(\mathbf{BC}) + \mathbf{W}_i(\boldsymbol{\beta}\boldsymbol{\gamma}) + \mathbf{W}_i(\boldsymbol{\eta}\boldsymbol{\xi}) \right) \right. \right. \\
&\quad \left. \left. + \boldsymbol{\gamma}_i \left( \boldsymbol{\eta}_i + (-1)^N \sum_{\{i;j,k\}} \left( \left( j+\frac{1}{2} \right) \boldsymbol{\eta}_{i+j+k} \boldsymbol{\xi}_j \partial \boldsymbol{\xi}_k - \frac{1}{4} \boldsymbol{\eta}_{i+j+k} D^a \boldsymbol{\xi}_j D^a \boldsymbol{\xi}_k \right) \right) \right] \right. \\
&\quad \left. + \sum_{\{0;i,j\}} \oint dZ_i \boldsymbol{\xi}_j \mathbf{W}_{i+j}^2 \left[ \mathbf{C}_i \left( \mathbf{W}_i^{(m)} + \frac{1}{2} \mathbf{W}_i(\mathbf{BC}) + \mathbf{W}_i(\boldsymbol{\beta}\boldsymbol{\gamma}) + \mathbf{W}_i(\boldsymbol{\eta}\boldsymbol{\xi}) \right) + \boldsymbol{\gamma}_i \boldsymbol{\eta}_i \right] \right],
\end{aligned}$$



$$Q_{BRST}(N, n) = \oint dZ \sum_{i=0}^{n-2} C_i \left( \mathbf{W}_i^{(m)} + \frac{1}{2} \mathbf{W}_i(BC) \right), \quad Q_t = \oint dZ \sum_{i=0}^{n-2} \gamma_i \eta_i.$$

This guarantees that the cohomology of the  $(N+1)$ -super  $w_n$  string coincides with the one of the  $N$ -super  $w_n$  string.

## 5. Discussions

In this paper, we have constructed a super  $w$  string hierarchy parametrized by two integers  $(N, n)$ .  $N$ -super  $w_n$  string can be derived as a broken phase of both the  $N$ -super  $w_{n+1}$  string and the  $(N+1)$ -super  $w_n$  string. This is a generalization of two hierarchies, superstring hierarchy<sup>[3]</sup> and  $w$  string hierarchy<sup>[5]</sup> and including them as subhierarchies  $(N, 2)$  and  $(0, n)$ , respectively.

We have proved that the cohomology of the broken phase  $N$ -super  $w_{n+1}$  string or  $(N+1)$ -super  $w_n$  string coincides with the one of the  $N$ -super  $w_n$  string. For the complete proof of equivalence, however, we must also show the cancellations of the zero modes, or equivalently the coincidence of amplitudes. This is nontrivial since the moduli spaces of two theories are different. Moreover, the integrand of the amplitudes are not BRST invariant in general but transformed to the total derivatives of the moduli. Such a investigation of global degrees of freedom was given in Ref. [1] for  $N = 1$  and 2 superstrings. It was also discussed for  $w$  string hierarchy by giving a generalization of the picture changing without studying the moduli space explicitly<sup>[5]</sup>. Similar consideration of the picture changing is possible for all the cases in the  $w$  string hierarchy. After similarity transformation, the Hilbert space of the broken phase string becomes direct product of the would-be equivalent string and a topological string. The difference of moduli spaces should provide the one of the topological theory. It is enough for the coincidence of amplitudes to show that all the topological string amplitudes are equal to one. This can be done, in principle, in a similar way to one using in Ref. [5]. We should give a geometrical consideration of the moduli space in any case.

As mentioned in Introduction, the broken-phase super  $w_{n+1}$  string is equivalent to the  $w_n$  string *in broken phase* since the  $w_n$  algebra is not a subalgebra of the  $w_{n+1}$  algebra. This is a common structure for the non-linear  $W_n$  algebra. If there is a hierarchy of the non-linear  $W$  string which has a similar structure of the  $w$  string hierarchy, it has no further structure since subalgebra of the non-linear  $W_n$  algebra is the only Virasoro algebra. However, the linear  $w_n$  algebra has the other subalgebras which we call  $w_{n/q}$  algebra generated by  $\{W_i; i = 0 \bmod q, i \leq n-2\}$ . We can consider the  $w_{n/q}$  string and construct it as a broken phase of the  $w_n$  string. For example, the  $w_4$  algebra can be realized by generators of  $w_{4/2}$  algebra  $\{W_0^{(m)}, W_2^{(m)}\}$  and NG fields  $(\beta_1, \gamma_1)$  with dimensions  $(3, -2)$  as

$$\begin{aligned} W_0 &= W_0^{(m)} - 3\beta_1\partial\gamma_1 - 2\partial\beta_1\gamma_1, \\ W_1 &= \beta_1 - 2W_2^{(m)}\partial\gamma_1 - \partial W_2^{(m)}\gamma_1, \\ W_2 &= W_2^{(m)}. \end{aligned}$$

It would be interesting to examine these symmetry breaking for studying general features of the  $w$  string. We hope to discuss this issue elsewhere.

There is a common feature for the generators in the broken phase strings. The additional (broken) generators starting from the linear term of  $\beta_i$  ( $\eta_i$ ), which leads the corresponding  $\gamma_i$  ( $\xi_i$ ) field transformed inhomogeneously by this transformation:  $\delta\gamma_i = \lambda_i + \dots$  ( $\delta\xi_i = \epsilon_i + \dots$ ). This means  $\gamma_i$  ( $\xi_i$ ) is the Nambu-Goldstone boson (fermion) and realizations constructed are two-dimensional analog of the non-linear realization<sup>[2]</sup>. Since these gauge symmetries are local, these NG fields become unphysical<sup>[9,2]</sup>. This should be understood in the Lagrangian formulation by making a model which exhibits the spontaneous symmetry breaking. Such a investigation in the Lagrangian formulation was given in Ref [10] and it was found the Lagrangian in the broken phase. It is interesting to find the Lagrangian in the general phase which derive the above broken-phase one by the spontaneous symmetry breaking.

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